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Star Products and Quantum Algebras

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I show explicitly that the star product on a triangular Poisson Lie group leads to a quantum algebra structure (triangular Hopf algebra) on the quantized enveloping algebra of the Lie algebra of the Lie group, and that equivalent star-products generate isomorphic quantum algebras.

1. INTRODUCTION

The development of the quantum inverse scattering method (QISM) (Faddeev, 1984) to investigate the integrable models of quantum field theory and statistical physics gives rise to some interesting algebraic constructions. These investigations allow one to select a special class of Hopf algebras now known as quantum groups and quantum algebras (Drinfeld, 1986; Jimbo, 1986). The nice R-matrix formulation of the quantum group theory (Faddeev *et al.*, 1989), based on the fundamental relation of QISM (the FRT relation), has given an additional impulse for these investigations. As is well known, quantum groups can be seen as noncommutative generalizations of topological spaces which have a group structure. Such a structure induces an Abelian Hopf algebra structure (Abe, 1980) on the algebra of smooth functions on the group. Quantum groups are defined then as non-Abelian Hopf algebras (Takhtajan, 1989). A way to generate them is to deform the Abelian product of the Hopf algebra of functions into a non-Abelian one (*-product), using the so-called quantization by deformation of star-quantization (Bayen *et al.*, 1978a, b; Flato *et al.*, 1975). This quantization technique gives a deformed product once it is assigned a Poisson bracket on the algebra of smooth functions. In order to obtain that the deformed algebra is a Hopf algebra, namely a quantum group, the starting group G has to be endowed with a

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Poisson–Lie structure. Finally, using the duality procedure, this quantization leads to the structure of a quantum algebra on the quantized enveloping algebra of the Lie algebra corresponding to the above Lie group G.

By contrast to our earlier work (Mansour, 1997, 1998a), the present paper shows explicitly that the star product on a triangular Poisson Lie group G leads to the structure of a quantum algebra (triangular Hopf algebra) on the quantized enveloping algebra of the Lie algebra corresponding to the Lie group G and that equivalents star-products generate isomorphic quantum algebras.

This paper is organized as follows. Section 2 reviews basic definitions of quantum algebras. Section 3 shows explicitly the main result: A star-product on a Poisson–Lie group leads to a quantum algebra structure on the quantized enveloping algebra of the Lie bialgebra corresponding to the Poisson–Lie group. The last section shows that equivalent star-products generate isomorphic quantum algebras.

2. QUANTUM ALGEBRAS

In this section we review some general aspects of the theory of quantum algebras, following mainly the presentation of Drinfeld (1986) and Faddeev *et al.* (1989). Quantum algebras are nontrivial examples of Hopf algebras. Let A denote such a Hopf algebra. It is equipped with a multiplication map m: $A \otimes A \rightarrow A$, a co-multiplication Δ: $A \rightarrow A \otimes A$, antipode S: $A \rightarrow A$, and counit ϵ: $A \rightarrow C$, where C is the field of complex numbers. We suppose that A contains the unit element 1, with $\Delta(1) = 1 \otimes 1$, $S(1) = 1$, $\epsilon(1) = 1$. These operations have the following properties:

$$m(a \otimes 1) = m(1 \otimes a) = a \tag{1}$$

$$m(m \otimes id) = m(id \otimes m) \tag{2}$$

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta \tag{3}$$

$$\Delta(ab) = \Delta(a)\Delta(b) \tag{4}$$

$$m(S \otimes id)\Delta(a) = m(id \otimes S)\Delta(a) = \epsilon(a)1 \tag{5}$$

$$S(ab) = S(b)S(a) \tag{6}$$

$$\Delta \circ S = (S \otimes S) \circ \Delta^{op} \tag{7}$$

$$(\epsilon \otimes id)\Delta = (id \otimes \epsilon)\Delta = id \tag{8}$$

$$\epsilon(ab) = \epsilon(a)\epsilon(b) \tag{9}$$

where a, $b \in A$, and P is the permutation operator $P(a \otimes b) = (b \otimes a)$. Equation (1) is the definition of the unit element, while Eqs. (2) and (3) are the associativity and the coassociativity of A, respectively. Equation (4) defines Δ to be a homomorphism of A to $A \otimes A$, and (5)–(9) are the defining properties of the counit and antipode.

Definition 1. A pair (A, R) consisting of a Hopf algebra A and an invertible element $R \in A \otimes A$ will be called a triangular Hopf algebra if

$$\Delta^{op} = R \Delta R^{-1} \tag{10}$$

$$(\Delta \otimes id) R = R^{13} R^{23} \tag{11}$$

$$(id \otimes \Delta) R = R^{13} R^{12} \tag{12}$$

$$R R^{21} = 1 \tag{13}$$

Here $\Delta^{op} = P \circ \Delta$ and the symbols R^{13}, R^{12}, R^{23}, R^{21} have the following meaning: If $R = \Sigma_i \, a_i \otimes b_i$, then

$$R^{13} = \sum_i a_i \otimes 1 \otimes b_i, \qquad R^{23} = \sum_i 1 \otimes a_i \otimes b_i$$

$$R^{12} = \sum_i a_i \otimes b_i \otimes 1, \qquad R^{21} = \sum_i b_i \otimes a_i$$

From (10) and (11) we deduce that R satisfies the quantum Yang–Baxter equation

$$R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12} \tag{14}$$

3. STAR-PRODUCT AND QUANTUM ALGEBRAS

Let G be a Lie group, and g its Lie algebra. The enveloping algebra of the Lie algebra g is defined to be the tensor algebra $T(\mathrm{g}) = \oplus_{k=0}^{\infty} \mathrm{g}^{\otimes k}$, modulo the ideal I in $T(\mathrm{g})$, generated by all elements in $T(\mathrm{g})$ of the form

$$x \otimes y - y \otimes x - [x, y] \tag{15}$$

Let 1 be the identity of the enveloping algebra; then the morphism of g into $U(\mathrm{g}) \otimes U(\mathrm{g})$ given by

$$x \to x \otimes 1 + 1 \otimes x \tag{16}$$

extends to a morphism

$$\Delta_0\colon \quad U(\mathrm{g}) \to U(\mathrm{g}) \otimes U(\mathrm{g}) \tag{17}$$

The antipode of the enveloping algebra is defined as a bijective map

$$S_0\colon \quad U(\mathrm{g}) \to U(\mathrm{g}) \tag{18}$$

such that for any $x \in$ g we have

$$S_0(x) = -x \tag{19}$$

Now let $r \in$ (g $\otimes$ g) be a solution of the classical Yang–Baxter equation

$$[\![r, r]\!] = 0 \tag{20}$$

where the Schouten bracket is defined as follows:

$$[\![r, r]\!] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]$$

Then the Lie bialgebra structure (g, $\delta(r)$) on g is given by the algebra 1-cocycle

$$\begin{aligned} \varphi\colon \quad & \mathrm{g} \to \mathrm{g} \otimes \mathrm{g} \\ & x \mapsto (ad_x \otimes 1 + 1 \otimes ad_x) r \end{aligned} \tag{21}$$

where ad_x stands for the adjoint representation and the Poisson–Lie structure on Lie group (G, r) is given by

$$\{\phi, \psi\} = \sum_{i,j} r^{ij}(X_i^r(\phi)X_j^r(\psi) - X_i^l(\phi)X_j^l(\psi)) \tag{22}$$

where $X_i^r = (R_g)_* X_i$ and $X_i^l = (L_g)_* X_i$ are the right and left vector fields, respectively, on the group G, (X_i) is a basis of g, $(R_g)_*$, $(L_g)_*$ are the derivative mappings corresponding to the right and left translation, respectively.

If we denote by $R(G)$ [$L(G)$] the set of all right (left)-invariant vector fields on (G, r), then using elementary properties of derivative mappings one can show that each of $L(G)$ and $R(G)$ is a vector space with a bracket operation that satisfies the Jacobi identity. Since every element of $L(G)$ or $R(G)$ is completely determined by its value at the identity element of (G, r), it follows that $L(G)$ and $R(G)$ are isomorphic to the Lie algebra [the tangent space to (G, r) at the identity (e)].

Such morphisms can be extended respectively to algebra morphisms

$$\begin{aligned} U(\mathrm{g}) &\to D^l(G) \\ A &\mapsto A^l \end{aligned} \tag{23}$$

$$\begin{aligned} U(\mathrm{g}) &\to D^r(G) \\ A &\mapsto A^r \end{aligned} \tag{24}$$

where $D^l(G)$ and $D^r(G)$, are respectively, the algebra of left-invariant differential operators and the algebra of right-invariant differential operators on the Poisson–Lie group (G, r), such that if A is given by the product $A = (x_1 \ldots x_N)$, then A^l and A^r are respectively given by

$$A^l = (x_1 \ldots x_N)^l = x_1^l \ldots x_N^l \tag{25}$$

and

$$A^r = (x_1 \ldots x_N)^r = (x_1)^r \ldots (x_N)^r \tag{26}$$

This implies that the action of $U(\mathrm{g})$ on $\mathrm{F}(G)$ [the space of smooth functions on the Poisson–Lie group (G, r)] will be given by

$$\langle X, Y^l(\phi)\rangle = \langle XY, \phi\rangle \tag{27}$$

and

$$\langle X, Y^r(\phi)\rangle = \langle S_0(Y)X, \phi\rangle \tag{28}$$

Now we give the following definition (Moreno and Valero, 1992).

Definition 2. A star-product on the Poisson–Lie group is defined as a bilinear map

$$\mathrm{F}(G) \times \mathrm{F}(G) \to \mathrm{F}(G)[[h]]$$

$$(\phi, \psi) \mapsto \phi * \psi = \sum_j h^j C_j(\phi, \psi) \tag{29}$$

such that:

(i) When the above map is extended to $\mathrm{F}(G)[[h]]$, it is formally associative

$$(\phi * \psi) * \chi = \phi * (\psi * \chi) \tag{30}$$

(ii) $C_0(\phi, \psi) = \phi \cdot \psi = \psi \cdot \phi$.

(iii) $C_1(\phi, \psi) = \{\phi, \psi\}$.

(iv) The two cochains $C_k(\phi, \psi)$ are bidifferential operators on $\mathrm{F}(G)$.

In this definition the Hopf algebra $\mathrm{F}(G)[[h]]$, with a new product $*$ and an unchanged coproduct is considered to be a topological Hopf algebra. We recall that deformations with unchanged coproduct are called preferred deformations (Gestenhaber, 1964; Gestenhaber and Schack, 1992). This condition is imposed on quantization because of the invariance property of the Poisson–Lie group bracket

$$\Delta(\{\phi, \psi\}) = \{\Delta(\phi), \Delta(\psi)\}$$

It is therefore natural to impose the same compatibility condition on the star-product with respect to the coproduct of $\mathrm{F}(G)$, i.e.,

$$\Delta(\phi * \psi) = (\Delta(\phi) * \Delta(\psi)) \tag{31}$$

is satisfied. The star-product on the right side is canonically defined on $\mathrm{F}(G) \otimes \mathrm{F}(G)$ by

$$(\phi \otimes \psi) * (\phi' \otimes \psi') = (\phi * \phi') \otimes (\psi * \psi') \tag{32}$$

Remark. If all C_k are a left (right)-invariant bidifferential operators, then the corresponding star-product is called a left (right)-invariant one.

Definition 3. Two star-products $*_1$ and $*_2$ defined on F(G) are said to be formally equivalent if there exists a series

$$T = id + \sum_{i=1}^{\infty} h^i T_i \tag{33}$$

where the T_i are differential operators, such that

$$T(\phi *_1 \psi) = T(\phi) *_2 T(\psi) \tag{34}$$

Thanks to the morphisms (23), (24), if C_i is a left-invariant two-cochain, then there is an $F_i \in U(\mathrm{g}) \otimes U(\mathrm{g})$ such that

$$C_i^l(\phi, \psi) = F_i^l(\phi \otimes \psi) \tag{35}$$

and similarly for the right-invariant two-cochain there exists an element $H_i \in U(\mathrm{g}) \otimes U(\mathrm{g})$ such that

$$C_j^r(\phi, \psi) = H_j^r(\phi \otimes \psi) \tag{36}$$

If we introduce the two elements of $U(\mathrm{g}) \otimes U(\mathrm{g})[[h]]$

$$F = 1 + \sum_{i\geq 1} F_i h^i$$

$$H = 1 + \sum_{j\geq 1} H_j h^j$$

then we obtain the following result:

Proposition 1. The associativity of the left-invariant star-product implies

$$(\Delta_0 \otimes id)F\cdot(F \otimes 1) = (1 \otimes \Delta_0)F\cdot(1 \otimes F) \tag{37}$$

and the associativity of the right-invariant star-product leads to the following equality:

$$(S_0^{\otimes 2}(H) \otimes 1)\cdot(\Delta_0 \otimes id)S_0^{\otimes 2}(H) = (1 \otimes S_0^{\otimes 2}(H))\cdot(1 \otimes \Delta_0)S_0^{\otimes 2}(H) \tag{38}$$

Proof. Writing the right-invariant star-product as

$$(\phi *^r \psi) = \mu(H^r(\phi \otimes \psi)) \tag{39}$$

where μ is the usual mutiplication on the algebra of smooth functions on the group and $H = 1 + 1/2hr + \Sigma_{i\geq 2}\, H_i h^i$, then for any element X in the enveloping algebra, we have

$$
\begin{aligned}
&\langle X, \phi *^r (\psi *^r \chi)\rangle \\
&\quad = \langle X, \mu(id \otimes \mu)((id \otimes \Delta_0)H^r \cdot H^r_{23}(\phi \otimes \psi \otimes \chi))\rangle \\
&\quad = \langle (id \otimes \Delta_0)\Delta_0(X), (id \otimes \Delta_0)H^r \cdot H^r_{23}(\phi \otimes \psi \otimes \chi)\rangle \\
&\quad = \langle (1 \otimes (S_0^{\otimes 2})H)(id \otimes \Delta_0)((S_0^{\otimes 2})H)(id \otimes \Delta_0)\Delta_0(X), (\phi \otimes \psi \otimes \chi)\rangle
\end{aligned} \tag{40}
$$

Similarly we find that

$$
\begin{aligned}
&\langle X, (\phi *^r \psi) *^r \chi\rangle \\
&\quad = \langle ((S_0^{\otimes 2})H \otimes 1)(\Delta_0 \otimes id)((S_0^{\otimes 2})H)(\Delta_0 \otimes id)\Delta_0(X), (\phi \otimes \psi \otimes \chi)\rangle
\end{aligned} \tag{41}
$$

So, comparing (40) and (41), we obtain the result (38); the same proof is valid for the left-invariant one.

Proposition 2. Assume that F is a left-invariant star product on the group G; then $S_0^{\otimes 2}\,(F)$ is a right-invariant star product on the group G.

Proof. By applying the operator $(S_0 \otimes S_0 \otimes S_0)$ to Eq. (37) and using the fact that $(S_0 \otimes S_0) \circ \Delta_0 = \Delta_0^{op} \circ S_0$, we find obviously Eq. (38).

We define the star-product on the Poisson–Lie group by the following expression:

$$\phi * \psi = \mu((S_0^{\otimes 2})^{-1}(F^{-1})^r \cdot F^l(\phi \otimes \psi)) \tag{42}$$

In fact, the product defined in this way is associative:

$$
\begin{aligned}
&(\phi * \psi) * \chi \\
&\quad = \mu((S_0^{\otimes 2})^{-1}(F^{-1})^r \cdot F^l(\mu((S_0^{\otimes 2})^{-1}(F^{-1})^r \cdot F^l(\phi \otimes \psi)) \otimes \chi)) \\
&\quad = \mu(\mu \otimes id)((\Delta_0 \otimes 1)((S_0^{\otimes 2})^{-1}(F^{-1})^r) \cdot (\Delta_0 \otimes 1)F^l \\
&\qquad \cdot ((S_0^{\otimes 2})^{-1}(F^{-1})^r \otimes 1) \cdot (F^l \otimes 1)(\phi \otimes \psi \otimes \chi)) \\
&\quad = \mu(\mu \otimes id)((\Delta_0 \otimes id)((S_0^{\otimes 2})^{-1}(F^{-1})^r) \cdot ((S_0^{\otimes 2})^{-1}(F^{-1})^r \otimes 1) \\
&\qquad \cdot (\Delta_0 \otimes id)F^l \cdot (F^l \otimes 1)(\phi \otimes \psi \otimes \chi)) \\
&\quad = \mu(\mu \otimes id)((id \otimes \Delta_0)((S_0^{\otimes 2})^{-1}(F^{-1})^r) \cdot (1 \otimes (S_0^{\otimes 2})^{-1}(F^{-1})^r) \\
&\qquad \cdot (id \otimes \Delta)F^l \cdot (1 \otimes F^l)(\phi \otimes \psi \otimes \chi))
\end{aligned}
$$

$$
\begin{aligned}
&= \mu(id \otimes \mu)((id \otimes \Delta_0)((S_0^{\otimes 2})^{-1}(F^{-1})^r)\cdot(1 \otimes (S_0^{\otimes 2})^{-1}(F^{-1})^r) \\
&\quad \cdot(id \otimes \Delta_0)F^l\cdot(1 \otimes F^l)(\phi \otimes \psi \otimes \chi)) \\
&= \mu(id \otimes \mu)((id \otimes \Delta_0)((S_0^{\otimes 2})^{-1}(F^{-1})^r)\cdot(id \otimes \Delta_0)F^l \\
&\quad \cdot(1 \otimes (S_0^{\otimes 2})^{-1}(F^{-1})^r)\cdot(1 \otimes F^l)(\phi \otimes \psi \otimes \chi)) \\
&= \mu((S_0^{\otimes 2})^{-1}(F^{-1})^r\cdot F^l\cdot(\phi \otimes \mu((S_0^{\otimes 2})^{-1}(F^{-1})^r\cdot F^l(\psi \otimes \chi))) \\
&= \phi * (\psi * \chi) \qquad (43)
\end{aligned}
$$

For the compatibility relation, the proof is given by Moreno and Valero (1992). Following Mansour (1998b), a star-product defines a deformation of a quotient algebra $F_e(G)$ defined as the set of elements of $F(G)$ in a neighborhood containing the identity of G modulo the following relation of equivalence:

$$\phi \sim \psi \qquad \text{if} \quad \langle X,\phi - \psi\rangle = 0 \qquad \text{for any} \quad X \in U(g)$$

where $\langle\ ,\ \rangle$ is the pairing between $F_e(G)$ and $U(g)$.

Let us recall that two bialgebras U, A are said to be in duality if there exists a doubly nondegenerate bilinear form

$$\langle\ ,\ \rangle\colon \quad U \times A \to C, \qquad \langle\ ,\ \rangle\colon \quad (u, a) \to \langle u, a\rangle, \qquad u \in U, \quad a \in A$$

such that for any u, $v \in U$ and a, $b \in A$

$$
\begin{aligned}
\langle u, ab\rangle &= \langle \Delta_U(u), a \otimes b\rangle \\
\langle uv, a\rangle &= \langle u \otimes v, \Delta_A(a)\rangle \\
\langle 1_U, a\rangle &= \epsilon_A(a), \qquad \langle u, 1_U\rangle = \epsilon_U(u)
\end{aligned}
$$

The duality between bialgebras can be used to obtain an unknown algebra from a known one if the two are in duality. So the deformation we talk about is a deformation of the $F_e(G)$ as a bialgebra; this allows us to provide by the duality the deformed algebra $F_e^*(G)[[h]]$ where $F_e^*(G)$ is the set of distributions on G with support at the the unit element (e). Then using the fact that the set of distributions on G with support at the identity element is the enveloping algebra of the Lie algebra of the Lie group, we deduce that a star-product provides a deformation of the enveloping algebra.

The quantized enveloping algebra $U(g)[[h]]$ is endowed with the structure of a Hopf algebra where the multiplication algebra is the ordinary convolution on $F_e^*(G)$ and the coproduct Δ_F is given by (Mansour, 1998b)

$$\langle \Delta_F(X), \phi \otimes \psi\rangle = \langle X, \phi * \psi\rangle \qquad (44)$$

for all ϕ, $\psi \in F_e(G)$ and $X \in U(g)$.

In fact, using Eq. (27) and (28) we obtain

$$\begin{aligned}\langle \Delta_F(X), \phi \otimes \psi \rangle &= \langle X, \mu((S_0^{\otimes 2})^{-1}(F^{-1})^r \cdot F^l(\phi \otimes \psi)) \rangle \\ &= \langle \Delta_0(X), (S_0^{\otimes 2})^{-1}(F^{-1})^r \cdot F^l(\phi \otimes \psi) \rangle \\ &= \langle F^{-1} \cdot \Delta_0(X) \cdot F, (\phi \otimes \psi) \rangle \end{aligned} \tag{45}$$

Thus

$$\Delta_F(X) = F^{-1} \cdot \Delta_0(X) \cdot F \tag{46}$$

The associativity of the star-product and Eq. (44) imply that the twisted coproduct Δ_F is coassociative, i.e.,

$$(\Delta_F \otimes id)\Delta_F = (id \otimes \Delta_F)\Delta_F$$

For the antipode of the quantized enveloping algebra, we recall first that the antipode S_0 of $U(g)$ satisfies the equation

$$m(S_0 \otimes id)\Delta_0(X) = m(id \otimes S_0)\Delta_0(X) = \varepsilon(X)1 \tag{47}$$

where m is the usual multiplication on the enveloping algebra $U(g)$. We can split F and F^{-1} respectively as

$$F = \sum_k a_k \otimes b_k, \qquad F^{-1} = \sum_k c_k \otimes d_k$$

and set $u = m(id \otimes S_0)(F^{-1})$. It is an invertible element of $U(g)[[h]]$; then we can easily show that the antipode of the quantized enveloping algebra $U(g)[[h]]$ is given by

$$S_F(X) = u \cdot S_0(X) \cdot u^{-1} \tag{48}$$

where $u^{-1} = m(S_0 \otimes id)F$.

In fact,

$$\begin{aligned} m(S_F \otimes id)\Delta_F(X) &= m(uS_0u^{-1} \otimes id)(F^{-1}\Delta_0(X)F) \\ &= \sum_{i,j,k} uS_0(a_i)S_0(X'_k)S_0(c_j)u^{-1}d_j X''_k b_i \end{aligned} \tag{49}$$

with $\Delta_0(X) = \Sigma_k X'_k \otimes X''_k$

Owing to the fact that S_0 satisfies Eq. (47) and that

$$\sum_j S_0(c_j)u^{-1}d_j = m(S_0 \otimes id)(F \cdot F^{-1}) = 1 \tag{50}$$

we obtain

$$m(S_F \otimes id)\Delta_F(X) = \sum_i uS_0(a_i)b_i \epsilon(X)1 = \epsilon(X)1 \tag{51}$$

Similarly, we can prove that

$$m(id \otimes S_F)\, \Delta_F(X) = \epsilon(X)1 \tag{52}$$

In other words, the topological Hopf algebra structure on $F(G)[[h]]$ is given by the following antipode:

$$S_h(f) = S((S_0^{-1}(u))^r(u^{-1})^l f)$$

The proof is obvious by using Eqs. (27), (28). Now if we introduce the following element defined by Drinfeld (1983)

$$R_F = F_{21}^{-1}\cdot F \tag{53}$$

then we can easily show that R_F defines a quasitriangular structure on the quantized enveloping algebra $U(g)[[h]]$.

In fact, using polynomial notation (Moreno and Valero, 1992), we obtain

$$\begin{aligned}
(\Delta_F \otimes id)R_F &= (F^{-1} \otimes 1)(\Delta \otimes id)R_F\,(F \otimes 1)\\
&= F^{-1}(x, y)F^{-1}(z, x + y)F(x + y, z)\; F(x, y)\\
&= F^{-1}(x, y)F^{-1}(z, x + y)F(x, y + z)\; F(y, z)\\
&= F^{-1}(x, y)F^{-1}(z, x + y)F(x, y + z)\; F(z, y)F^{-1}(z, y)F(y, z)\\
&= F^{-1}(x, y)F^{-1}(z, x + y)F(x, y + z)\; F(z, y)R_F(y, z)\\
&= F^{-1}(x, y)F^{-1}(z, x + y)F(x + z, y)\; F(x, z)R_F(y, z)\\
&= F^{-1}(x, y)F^{-1}(z, x + y)F(x + z, y)\; F(z, x)R_F(x, z)R_F(y, z)\\
&= R_F(x, z)R_F(y, z)
\end{aligned} \tag{54}$$

Thus,

$$(\Delta_F \otimes id)R_F = (R_F)_{13}\cdot(R_F)_{23} \tag{55}$$

where we have used the definition (53) in the first, sixth, and seventh equalities and the relation (37) written in polynomial notation for the remaining ones.

Similarly, we obtain

$$(id \otimes \Delta_F)R_F = (R_F)_{13}\cdot(R_F)_{12} \tag{56}$$

From the fact that $\phi * 1 = 1 * \phi = \phi$ for all $\phi \in \mathrm{F}_e(G)$ we deduce that

$$(id \otimes \varepsilon)F = (\varepsilon \otimes id)F = 1 \tag{57}$$

Consequently

$$(\varepsilon \otimes id)(R_F) = (id \otimes \varepsilon)(R_F) = 1 \tag{58}$$

and from the definition (53) we deduce that

$$(R_F)_{21}\cdot R_F = 1 \tag{59}$$

Now using again the expression (53), we obtain that

$$\begin{aligned}(\Delta_F)^{op} &= P(\Delta_F)\\ &= P(F^{-1})\cdot\Delta_0\cdot\, P(F)\\ &= P(F^{-1})\cdot F\cdot\Delta_F\,\cdot F^{-1}\cdot P(F)\end{aligned} \tag{60}$$

Then

$$(\Delta_F)^{op} = R_F\cdot\Delta_F\cdot(R_F)^{-1} \tag{61}$$

From (54) and (60) we show that R_F satisfies the quantum Yang–Baxter equation

$$(R_F)_{12}\cdot(R_F)_{13}\,\cdot(R_F)_{23} = (R_F)_{23}\cdot\,(R_F)_{13}\cdot(R_F)_{12} \tag{62}$$

Then we have establiched that a star-product on a Poisson–Lie group (G, r) leads to a quantum algebra $(Ug[[h]], \Delta_F, R_F, S_F)$ where $F = 1 + \frac{1}{2}hr + \Sigma_{i\geq 2}\, F_i h^i$ and $R_F = 1 + hr + \ldots$; conversely, we have the following result:

Theorem 1. Let $(Ug[[h]], \Delta, R, S)$ be a triangular Hopf algebra; then it can be obtained by a star-product on the connected and simply connected Poisson–Lie group (G, r) corresponding to the Lie algebra g, where $R = 1 + hr + \ldots.$

4. EQUIVALENTS STAR-PRODUCTS ON A POISSON–LIE GROUP

Let F and $\bar{F}$ be two star-products, i.e., two elements of the Hopf algebra $U(g)[[h]]$, and let $A = (U((g)[[h]], \Delta_F, R_F, S_F)$ and $\bar{A} = (U((g)[[h]], \Delta_{\bar{F}}, R_{\bar{F}}, S_{\bar{F}})$ be the resulting quantum algebras, where

$$\Delta_F = F\cdot\Delta_0\cdot F^{-1}, \qquad R_F = F_{21}^{-1}\cdot F \tag{63}$$

$$\Delta_{\bar{F}} = \bar{F}\cdot\Delta_0\cdot\bar{F}^{-1}, \qquad R_{\bar{F}} = \bar{F}_{21}^{-1}\cdot\bar{F} \tag{64}$$

Then it is easily seen that $\bar{A}$ can be obtained from A by applying the twist $\hat{F} = F^{-1}\cdot\bar{F}$. In fact

$$\Delta_{\bar{F}} = \hat{F}\cdot\Delta_F\cdot\hat{F}^{-1} \tag{65}$$

and

$$R_{\bar{F}} = \hat{F}_{21}^{-1}\cdot R_F\cdot\hat{F} \tag{66}$$

If the two star-products are equivalent, i.e., the corresponding elements F and $\bar{F}$ are related by the expression

$$\bar{F} = \Delta_0(E^{-1})\cdot F\cdot(E \otimes E) \quad (67)$$

for some invertible element E of $U(\mathrm{g})[[h]]$, then the coproduct $\Delta_{\bar{F}}$ can be rewritten as

$$\Delta_{\bar{F}}(X) = (E^{-1} \otimes E^{-1})\Delta_F(E\cdot X\cdot E^{-1})\cdot(E \otimes E) \quad (68)$$

In fact

$$\begin{aligned} \Delta_{\bar{F}}(X) &= \bar{F}^{-1}\cdot\Delta_0(X)\cdot\bar{F} \\ &= (E^{-1} \otimes E^{-1})\cdot F^{-1}\cdot\Delta_0(E)\Delta_0(X)\Delta_0(E^{-1})\cdot F\cdot(E \otimes E) \\ &= (E^{-1} \otimes E^{-1})\cdot F^{-1}\cdot\Delta_0(E\cdot X\cdot E^{-1})\cdot F\cdot(E \otimes E) \\ &= (E^{-1} \otimes E^{-1})\cdot\Delta_F(E\cdot X\cdot E^{-1})\cdot(E \otimes E) \end{aligned} \quad (69)$$

and the two twisted antipodes are related by the following expression:

$$S_{\bar{F}} = E^{-1}S_0(E^{-1})\cdot S_F\cdot S_0(E)\cdot E \quad (70)$$

In fact,

$$\begin{aligned} S_{\bar{F}} &= m(id \otimes S_0)(\bar{F}^{-1})S_0(X)m\,(S_0 \otimes id)(\bar{F}) \\ &= m(id \otimes S_0)((E^{-1} \otimes E^{-1})F^{-1}\Delta(E)) \\ &\quad \times S_0(X)m(S_0 \otimes id)(\Delta(E^{-1})F(E \otimes E)) \\ &= m(id \otimes S_0)(E^{-1} \otimes E^{-1})\cdot m(id \otimes S_0)F^{-1}\ m(id \otimes S_0)\Delta(E)\ S_0(X) \\ &\quad \times m(S_0 \otimes id)(\Delta(E^{-1}))m(S_0 \otimes id)(F)m(S_0 \otimes id)(E \otimes E)) \\ &= E^{-1}S_0(E^{-1})\ uS_0(X)u^{-1}S_0(E)S_0(\mathrm{E}) \\ &= E^{-1}S_0(E^{-1})\cdot S_F(X)\cdot S_0(E)\cdot E \end{aligned}$$

Similarly, the tringular structures are related by

$$R_{\bar{F}} = (E^{-1} \otimes E^{-1})\cdot R_F\cdot(E \otimes E) \quad (71)$$

In fact, using again the polynomial notation, we have

$$\begin{aligned} R_{\bar{F}}(x, y) &= \bar{F}^{-1}(y, x)\cdot\bar{F}(x, y) \\ &= E^{-1}(y)E^{-1}(x)F^{-1}(y, x)E(y + x)E^{-1}(x + y)F(x, y)\cdot\ E(x)E(y) \\ &= E^{-1}(y)E^{-1}(x)F^{-1}(y, x)F(x, y)\cdot E(x)E(y) \\ &= (E^{-1} \otimes E^{-1})(x, y)R_F(x, y)\cdot(E \otimes E)(x, y) \\ &= ((E^{-1} \otimes E^{-1})\cdot R_F\cdot(E \otimes E)(x, y) \end{aligned}$$

So, the induced isomorphism maps the triangular structures as well. This says that the process of quantization deformation can only give a genuinely new triangular quantum group if the two-cocycle F corresponding to the star-product is cohomologically [relative to the Hopf algebra cohomology (Majid, 1995)] nontrivial; for example, if the second group of cohomology for the Hopf algebra $U(\mathrm{g})[[h]]$ vanishes, then all star-products on the connected and simply connected Poisson–Lie group corresponding to the Lie bialgebra g are equivalents.

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